

Dilatation structures with the Radon-Nikodym property

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Introduction

The notion of a dilatation structure stemmed out from my efforts to understand basic results in sub-Riemannian geometry, especially the last section of the paper by Bellaïche [2] and the intrinsic point of view of Gromov [5].

In these papers, as in other articles devoted to sub-Riemannian geometry, fundamental results admitting an intrinsic formulation were proved using differential geometry tools, which are in my opinion not intrinsic to sub-Riemannian geometry.

Therefore I tried to find a self-contained frame in which sub-Riemannian geometry would be a model, if we use the same manner of speaking as in the case of hyperbolic geometry (with its self-contained collection of axioms) and the Poincaré disk as a model of hyperbolic geometry.

An outcome of this effort are the notions of a dilatation structure and a pair of dilatation structures, one looking down to another. To the first notion are dedicated the papers [3], [4] (the second paper treating about a "linear" version of a generalized dilatation structure, corresponding to Carnot groups or more general contractible groups).

As it seems now, dilatation structures are a valuable notion by itself, with possible field of application strictly containing sub-Riemannian geometry, but also ultrametric spaces or contractible groups. A dilatation structure encodes the approximate self-similarity of a metric space and it induces non associative but approximately associative operations on the metric space, as well as a tangent bundle (in the metric sense) with group operations in each fiber (tangent space to a point).

In this paper I explain what is a pair of dilatation structures, one looking down to another, see definition 3.5. Such a pair of dilatation structures leads to the intrinsic definition of a distribution as a field of topological filters, definition 3.6.

To any pair of dilatation structures there is an associated notion of differentiability which generalizes the Pansu differentiability [8]. This allows the introduction of

the Radon-Nikodym property for dilatation structures, which is the straightforward generalization of the Radon-Nikodym property for Banach spaces.

After an introducing section about length metric spaces and metric derivatives, is proved in theorem 3.4 that for a dilatation structure with the Radon-Nikodym property the length of absolutely continuous curves expresses as an integral of the norms of the tangents to the curve, as in Riemannian geometry.

Further it is shown that Radon-Nikodym property transfers from any "upper" dilatation structure looking down to a "lower" dilatation structure, theorem 3.7. In my opinion this result explains intrinsically the fact that absolutely continuous curves in regular sub-Riemannian manifolds are derivable almost everywhere, as proved by Margulis, Mostow [7], Pansu [8] (for Carnot groups) or Vodopyanov [10].

The subject of application of these results for regular sub-Riemannian manifold will be left for a future paper, due to the unavoidable accumulation of technical estimates which are needed.

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1	Notations	3
2	Length and metric derivatives	3
3	The Radon-Nikodym property	6
3.1	Length formula from Radon-Nikodym property	8
3.2	A dilatation structure looking down to another	8
3.3	Transfer of Radon-Nikodym property	10
4	Appendix: Dilatation structures	12
4.1	The axioms of a dilatation structure	12
4.2	Tangent bundle of a dilatation structure	14
4.3	Equivalent dilatation structures	15
4.4	Differentiable functions	16

1 Notations

Let Γ be a topological separated commutative group endowed with a continuous group morphism

$$\nu : \Gamma \rightarrow (0, +\infty)$$

with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of Γ is denoted by 1. We use the multiplicative notation for the operation in Γ .

The morphism ν defines an invariant topological filter on Γ (equivalently, an end). Indeed, this is the filter generated by the open sets $\nu^{-1}(0, a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \rightarrow 0$ for $\nu(\varepsilon) \in (0, +\infty) \rightarrow 0$.

The set $\Gamma_1 = \nu^{-1}(0, 1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$. On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on Γ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in Γ .

The space (X, d) is a complete, locally compact metric space. For any $r > 0$ and any $x \in X$ we denote by $B(x, r)$ the open ball of center x and radius r in the metric space X .

By $\mathcal{O}(\varepsilon)$ we mean a positive function $f : \Gamma \rightarrow [0, +\infty)$ such that $\lim_{\varepsilon \rightarrow 0} f(\nu(\varepsilon)) = 0$.

2 Length and metric derivatives

For a detailed introduction into the subject see for example [1], chapter 1.

Definition 2.1 *The (upper) dilatation of a map $f : X \rightarrow Y$ between metric spaces, in a point $u \in Y$ is*

$$Lip(f)(u) = \limsup_{\varepsilon \rightarrow 0} \sup \left\{ \frac{d_Y(f(v), f(w))}{d_X(v, w)} : v \neq w, v, w \in B(u, \varepsilon) \right\}$$

In the particular case of a derivable function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ the upper dilatation is $Lip(f)(t) = |\dot{f}(t)|$. For any Lipschitz function $f : X \rightarrow Y$ and for any $x \in X$ we have the obvious relation:

$$Lip(f)(x) \leq Lip(f) .$$

A curve is a continuous function $f : [a, b] \rightarrow X$. The image of a curve is called path. Length measures paths. Therefore length does not depends on the reparametrisation of the path and it is additive with respect to concatenation of paths.

In a metric space (X, d) one can measure the length of curves in several ways.

Definition 2.2 *The length of a curve with L^1 dilatation $f : [a, b] \rightarrow X$ is*

$$L(f) = \int_a^b Lip(f)(t) dt$$

A different way to define a length of a curve is to consider its variation.

Definition 2.3 *The curve f has bounded variation if the quantity*

$$Var(f) = \sup \left\{ \sum_{i=0}^n d(f(t_i), f(t_{i+1})) : a = t_0 < t_1 < \dots < t_n < t_{n+1} = b \right\}$$

(called variation of f) is finite.

There is a third, more basic way to introduce the length of a curve in a metric space.

Definition 2.4 *The length of the path $A = f([a, b])$ is the one-dimensional Hausdorff measure of the path. The definition is the following:*

$$l(A) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i \in I} \text{diam } E_i : \text{diam } E_i < \delta, \quad A \subset \bigcup_{i \in I} E_i \right\}$$

The definitions are not equivalent. The variation $Var(f)$ of a curve f and the length of a path $L(f)$ do not agree in general. Consider for example: $f : [-1, 1] \rightarrow \mathbb{R}^2$, $f(t) = (t, \text{sign}(t))$. We have $Var(f) = 4$ and $L(f([-1, 1])) = 2$. Another example: the Cantor staircase function is continuous, but not Lipschitz. It has variation equal to 1 and length of the graph equal to 2.

Nevertheless, for Lipschitz functions, the first two definitions agree. For injective Lipschitz functions (i.e. for simple Lipschitz curves) the last two definitions agree.

Theorem 2.5 *For each Lipschitz curve $f : [a, b] \rightarrow X$, we have $L(f) = Var(f)$.*

Theorem 2.6 *Suppose that $f : [a, b] \rightarrow X$ is a Lipschitz function and $A = f([a, b])$. Then $\mathcal{H}^1(A) \leq Var(f)$.*

If f is moreover injective then $\mathcal{H}^1(A) = Var(f)$.

An important tool used in the proof of the previous theorem is the geometrically obvious, but not straightforward to prove in this generality, Reparametrisation Theorem.

Theorem 2.7 *Any path $A \subset X$ with a Lipschitz parametrisation admits a reparametrisation $f : [a, b] \rightarrow A$ such that $Lip(f)(t) = 1$ for almost any $t \in [a, b]$.*

We shall denote by l_d the length functional, defined only on Lipschitz curves, induced by the distance d . The length induces a new distance d_l , say on any Lipschitz connected component of the space (X, d) . The distance d_l is given by:

$$d_l(x, y) = \inf \{l_d(f([a, b])) : f : [a, b] \rightarrow X \text{ Lipschitz , } \\ f(a) = x , f(b) = y\}$$

We have therefore two operators $d \mapsto l_d$ and $l \mapsto d_l$. This leads to the introduction of length metric spaces.

Definition 2.8 *A length metric space is a metric space (X, d) such that $d = d_l$.*

From theorem 2.5 we deduce that Lipschitz curves in complete length metric spaces are absolutely continuous. Indeed, here is the definition of an absolutely continuous curve (definition 1.1.1, chapter 1, [1]).

Definition 2.9 *Let (X, d) be a complete metric space. A curve $c : (a, b) \rightarrow X$ is absolutely continuous if there exists $m \in L^1((a, b))$ such that for any $a < s \leq t < b$ we have*

$$d(c(s), c(t)) \leq \int_s^t m(r) \, dr.$$

Such a function m is called an upper gradient of the curve c .

According to theorem 2.5, for a Lipschitz curve $c : [a, b] \rightarrow X$ in a complete length metric space such a function $m \in L^1((a, b))$ is the upper dilatation $Lip(c)$. More can be said about the expression of the upper dilatation. We need first to introduce the notion of metric derivative of a Lipschitz curve.

Definition 2.10 *A curve $c : (a, b) \rightarrow X$ is metrically derivable in $t \in (a, b)$ if the limit*

$$md(c)(t) = \lim_{s \rightarrow t} \frac{d(c(s), c(t))}{|s - t|}$$

exists and it is finite. In this case $md(c)(t)$ is called the metric derivative of c in t .

For the proof of the following theorem see [1], theorem 1.1.2, chapter 1.

Theorem 2.11 *Let (X, d) be a complete metric space and $c : (a, b) \rightarrow X$ be an absolutely continuous curve. Then c is metrically derivable for \mathcal{L}^1 -a.e. $t \in (a, b)$. Moreover the function $md(c)$ belongs to $L^1((a, b))$ and it is minimal in the following sense: $md(c)(t) \leq m(t)$ for \mathcal{L}^1 -a.e. $t \in (a, b)$, for each upper gradient m of the curve c .*

3 The Radon-Nikodym property

Definition 3.1 A dilatation structure (X, d, δ) has the Radon-Nikodym property if any Lipschitz curve $c : [a, b] \rightarrow (X, d)$ is derivable almost everywhere.

Example 3.1 For $(X, d) = (\mathbb{V}, d)$, a real, finite dimensional, normed vector space, with distance d induced by the norm, the (usual) dilatations δ_ε^x are given by:

$$\delta_\varepsilon^x y = x + \varepsilon(y - x)$$

Dilatations are defined everywhere. The group Γ is $(0, +\infty)$ and the function ν is the identity.

There are few things to check (see the appendix): axioms 0,1,2 are obviously true. For axiom A3, remark that for any $\varepsilon > 0$, $x, u, v \in X$ we have:

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = d(u, v) ,$$

therefore for any $x \in X$ we have $d^x = d$.

Finally, let us check the axiom A4. For any $\varepsilon > 0$ and $x, u, v \in X$ we have

$$\begin{aligned} \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v &= x + \varepsilon(u - x) + \frac{1}{\varepsilon} (x + \varepsilon(v - x) - x - \varepsilon(u - x)) = \\ &= x + \varepsilon(u - x) + v - u \end{aligned}$$

therefore this quantity converges to

$$x + v - u = x + (v - x) - (u - x)$$

as $\varepsilon \rightarrow 0$. The axiom A4 is verified.

This dilatation structure has the Radon-Nikodym property. \square

Example 3.2 Because dilatation structures are defined by local requirements, we can easily define dilatation structures on riemannian manifolds, using particular atlases of the manifold and the riemannian distance (infimum of length of curves joining two points). Note that any finite dimensional manifold can be endowed with a riemannian metric. This class of examples covers all dilatation structures used in differential geometry. The axiom A4 gives an operation of addition of vectors in the tangent space (compare with Bellaïche [2] last section). \square

Example 3.3 Take $X = \mathbb{R}^2$ with the euclidean distance d . For any $z \in \mathbb{C}$ of the form $z = 1 + i\theta$ we define dilatations

$$\delta_\varepsilon x = \varepsilon^z x .$$

It is easy to check that $(\mathbb{R}^2, d, \delta)$ is a dilatation structure, with dilatations

$$\delta_\varepsilon^x y = x + \delta_\varepsilon(y - x) .$$

Two such dilatation structures (constructed with the help of complex numbers $1 + i\theta$ and $1 + i\theta'$) are equivalent if and only if $\theta = \theta'$.

There are two other interesting properties of these dilatation structures. The first is that if $\theta \neq 0$ then there are no non trivial Lipschitz curves in X which are differentiable almost everywhere. It means that such dilatation structure does not have the Radon-Nikodym property.

The second property is that any holomorphic and Lipschitz function from X to X (holomorphic in the usual sense on $X = \mathbb{R}^2 = \mathbb{C}$) is differentiable almost everywhere, but there are Lipschitz functions from X to X which are not differentiable almost everywhere (suffices to take a C^∞ function from \mathbb{R}^2 to \mathbb{R}^2 which is not holomorphic). \square

The Radon-Nikodym property can be stated in two equivalent ways.

Proposition 3.2 *Let (X, d, δ) be a dilatation structure. Then the following are equivalent:*

- (a) (X, d, δ) has the Radon-Nikodym property;
- (b) any Lipschitz curve $c' : [a', b'] \rightarrow (X, d)$ admits a reparametrization $c : [a, b] \rightarrow (X, d)$ such that for almost every $t \in [a, b]$ there is $\dot{c}(t) \in U(c(t))$ such that

$$\frac{1}{\varepsilon} d(c(t + \varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \rightarrow 0$$

$$\frac{1}{\varepsilon} d(c(t - \varepsilon), \delta_\varepsilon^{c(t)} \text{inv}^{c(t)}(\dot{c}(t))) \rightarrow 0 \quad ;$$

- (c) any Lipschitz curve $c' : [a', b'] \rightarrow (X, d)$ admits a reparametrization $c : [a, b] \rightarrow (X, d)$ such that for almost every $t \in [a, b]$ there is a conical group morphism

$$\dot{c}(t) : \mathbb{R} \rightarrow T_{c(t)}X$$

such that for any $a \in \mathbb{R}$ we have

$$\frac{1}{\varepsilon} d(c(t + \varepsilon a), \delta_\varepsilon^{c(t)} \dot{c}(t)(a)) \rightarrow 0.$$

Proof. It is straightforward that a conical group morphism $f : \mathbb{R} \rightarrow (N, \delta)$ is defined by its value $f(1) \in N$. Indeed, for any $a > 0$ we have $f(a) = \delta_a f(1)$ and for any $a < 0$ we have $f(a) = \delta_a f(1)^{-1}$. From the morphism property we also deduce that

$$\delta v = \{ \delta_a v : a > 0, v = f(1) \text{ or } v = f(1)^{-1} \}$$

is a one parameter group and that for all $\alpha, \beta > 0$ we have

$$\delta_{\alpha+\beta} u = \delta_\alpha u \delta_\beta u \quad \square$$

Definition 3.3 In a conical group N we shall denote by $D(N)$ the set of all $u \in N$ with the property that $\varepsilon \in ((0, \infty), +) \mapsto \delta_\varepsilon u \in N$ is a morphism of semigroups .

$D(N)$ is always non empty, because it contains the neutral element of N . $D(N)$ is also a cone, with dilatations δ_ε , and a closed set.

We shall always identify a conical group morphism $f : \mathbb{R} \rightarrow N$ with its value $f(1) \in D(N)$.

3.1 Length formula from Radon-Nikodym property

Theorem 3.4 Let (X, d, δ) be a dilatation structure with the Radon-Nikodym property, over a complete length metric space (X, d) . Then for any Lipschitz curve $c : [a, b] \rightarrow X$ the length of $\gamma = c([a, b])$ is

$$L(\gamma) = \int_a^b d^{c(t)}(c(t), \dot{c}(t)) \, dt.$$

Proof. The upper dilatation of c in t is

$$Lip(c)(t) = \limsup_{\varepsilon \rightarrow 0} \sup \left\{ \frac{d(c(v), c(w))}{|v - w|} : v \neq w, |v - t|, |w - t| < \varepsilon \right\}.$$

From theorem 2.11 we deduce that for almost every $t \in (a, b)$ we have

$$Lip(c)(t) = \lim_{s \rightarrow t} \frac{d(c(s), c(t))}{|s - t|}.$$

If the dilatation structure has the Radon-Nikodym property then for almost every $t \in [a, b]$ there is $\dot{c}(t) \in D(T_{c(t)}X)$ such that

$$\frac{1}{\varepsilon} d(c(t + \varepsilon), \delta_\varepsilon^{c(t)} \dot{c}(t)) \rightarrow 0.$$

Therefore for almost every $t \in [a, b]$ we have

$$Lip(c)(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} d(c(t + \varepsilon), c(t)) = d^{c(t)}(c(t), \dot{c}(t)).$$

The formula for length follows from here. \square

3.2 A dilatation structure looking down to another

Consider two dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$. We explain here in which sense \mathcal{A} looks down at \mathcal{B} .

Definition 3.5 Given dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$, we write that $\mathcal{A} \geq \mathcal{B}$ if the following conditions are fulfilled:

- (a) the identity $id : (X, d_A) \rightarrow (X, d_B)$ is 1-Lipschitz,
- (b) the identity $id : (X, d_A) \rightarrow (X, d_B)$ is derivable everywhere and for any point $x \in X$ the derivative $D id(x)$ is a projector,
- (c) for any $x \in X$, any continuous curve $\varepsilon \in [0, 1) \mapsto z(\varepsilon) \in X$, such that $d_A^x(z(0), x) \leq 3/2$, if

$$\lim_{\varepsilon \rightarrow 0} \left(d_A^x(x, z(\varepsilon)) - \frac{1}{\varepsilon} d_B^x(x, \delta_\varepsilon^x z(\varepsilon)) \right) = 0$$

then $\lim_{\varepsilon \rightarrow 0} d_A^x(Q_\varepsilon^x z(\varepsilon), z(\varepsilon)) = 0$, where $Q_\varepsilon^x = \bar{\delta}_{\varepsilon-1}^x \delta_\varepsilon^x$.

We explain in more detail the meaning of this definition. Condition (a) says that for any $x, y \in X$ we have $d_B(x, y) \leq d_A(x, y)$. Condition (b) can be understood by using definition 4.10: for any $x \in X$ there exists a function $D id(x)$ defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that

$$\limsup_{\varepsilon \rightarrow 0} \left\{ \frac{1}{\varepsilon} \bar{d} \left(\delta_\varepsilon^x u, \bar{\delta}_\varepsilon^x D id(x)(u) \right) : d(x, u) \leq \varepsilon \right\} = 0. \quad (3.2.1)$$

From here we deduce that for any x and u such that $d_B(x, u)$ is sufficiently small

$$\lim_{\varepsilon \rightarrow 0} \bar{\delta}_{\varepsilon-1}^x \delta_\varepsilon^x(u) = D id(x)(u)$$

and the limit is uniform with respect to u .

The second part of the condition (b) states that

$$D id(x) D id(x) = D id(x).$$

In order to understand the condition (c) we need to introduce the following topological version of a distribution.

Definition 3.6 We denote by $TopD(x)$ the topological filter induced by the relatively open neighbourhoods of x in the closed ball $\{z \in X : d_A^x(x, z) \leq 2\}$, given by

$$F(x, \varepsilon, \lambda) = \left\{ z \in X : d_A^x(x, z) \leq 2, d_A^x(x, z) - \frac{1}{\varepsilon} d_B^x(x, \delta_\varepsilon^x z) \leq \lambda \right\}.$$

This filter is called the topological distribution associated with the pair of dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$, such that $\mathcal{A} \geq \mathcal{B}$.

With this notation we may rewrite the condition (c) definition 3.5 like this: let $z(\varepsilon)$ be a continuous curve such that (in the sense of topological filters)

$$\lim_{\varepsilon \rightarrow 0} z(\varepsilon) \in TopD(x).$$

Then $\lim_{\varepsilon \rightarrow 0} d_A^x(Q_\varepsilon^x z(\varepsilon), z(\varepsilon)) = 0$, where $Q_\varepsilon^x = \bar{\delta}_{\varepsilon-1}^x \delta_\varepsilon^x$. This means that the "size of the vertical part" of $z(\varepsilon)$, which is $d_A^x(Q_\varepsilon^x z(\varepsilon), z(\varepsilon))$, becomes arbitrarily small as $\varepsilon \rightarrow 0$.

3.3 Transfer of Radon-Nikodym property

Suppose that (X, d_A) and (X, d_B) are complete, locally compact, length metric spaces and that we have two dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$, such that $\mathcal{A} \geq \mathcal{B}$.

A sufficient condition to have (a) in definition 3.5 is the following (true in the case of sub-Riemannian manifolds):

(a') for any Lipschitz curve c , if $l_A(c) < +\infty$ then $l_B(c) = l_A(c)$. Here l_A and l_B denote the length functional associated to distance d_A , distance d_B respectively.

We prove here the following result concerning the transfer of Radon-Nikodym property.

Theorem 3.7 *Let (X, d_A) and (X, d_B) be complete, locally compact, length metric spaces. Suppose that we have two dilatation structures $\mathcal{A} = (X, d_A, \delta)$ and $\mathcal{B} = (X, d_B, \bar{\delta})$, such that $\mathcal{A} \geq \mathcal{B}$. Under the assumptions (a') and (b), (c), (d) from definition 3.5, if the dilatation structure $\mathcal{B} = (X, d_B, \bar{\delta})$ has the Radon-Nikodym property, then the dilatation structure $\mathcal{A} = (X, d_A, \delta)$ has the Radon-Nikodym property.*

Proof. Let $c : [0, 1] \rightarrow (X, d_A)$ be a Lipschitz curve. Because of hypothesis (a) it follows that $c : [0, 1] \rightarrow (X, d_B)$ is also Lipschitz. Moreover, we can reparametrize the curve c with the d_A length and so we can suppose that c is d_A 1-Lipschitz. Therefore we can suppose that c is d_B 1-Lipschitz.

The dilatation structure $\mathcal{B} = (X, d_B, \bar{\delta})$ has the Radon-Nikodym property. Then for almost any $t \in [0, 1]$ there is $\dot{c}(t)$ such that

$$\frac{1}{\varepsilon} d_B(c(t + \varepsilon), \bar{\delta}_\varepsilon^x \dot{c}(t)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (3.3.2)$$

$$\frac{1}{\varepsilon} d_B(c(t - \varepsilon), \bar{\delta}_\varepsilon^x \dot{c}(t)^{-1}) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (3.3.3)$$

Further we shall give only the half of the proof, namely we shall use only relation (3.3.2). To get a complete proof, one has to repeat the reasoning starting from (3.3.3).

Because c is d_A 1-Lipschitz, it follows that

$$d_A^{c(t)}(\delta_{\varepsilon^{-1}}^{c(t)} c(t + \varepsilon), \dot{c}(t)) \leq 2$$

for any $\varepsilon < \varepsilon(t) \in (0, +\infty)$. From the local compactness with respect to $d_A^{c(t)}$ we find that for any $t \in [0, 1]$ there is a sequence $(\varepsilon_h)_h \subset (0, +\infty)$, converging to 0 as $h \rightarrow \infty$, and $u(t) \in X$ such that:

$$\lim_{h \rightarrow \infty} \delta_{\varepsilon_h^{-1}}^{c(t)} c(t + \varepsilon_h) = u(t)$$

Use equation (3.3.2) to get that

$$\lim_{h \rightarrow \infty} \delta_{\varepsilon_h}^{c(t)} c(t + \varepsilon_h) = \dot{c}(t)$$

Re-write this latter equation as:

$$\lim_{h \rightarrow \infty} \bar{\delta}_{\varepsilon_h}^{c(t)} \delta_{\varepsilon_h}^{c(t)} \delta_{\varepsilon_h}^{c(t)} c(t + \varepsilon_h) = \dot{c}(t)$$

and use the first part of hypothesis (b) to get

$$D \operatorname{id}(c(t)) u(t) = \dot{c}(t)$$

But according to the second part of the hypothesis (b) the operator $D \operatorname{id}(c(t))$ is a projector, hence

$$D \operatorname{id}(c(t)) \dot{c}(t) = \dot{c}(t)$$

Because of the fact that the derivative commutes with dilatations we get the important fact that for any $\varepsilon > 0$

$$\delta_\varepsilon^{c(t)} \dot{c}(t) = \bar{\delta}_\varepsilon^{c(t)} \dot{c}(t) \quad (3.3.4)$$

We wish to prove

$$\frac{1}{\varepsilon} d_A(\delta_\varepsilon^{c(t)} \bar{\delta}_{\varepsilon^{-1}}^{c(t)} c(t + \varepsilon), c(t + \varepsilon)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (3.3.5)$$

Suppose that (3.3.5) is true. Then we would have

$$d_A^{c(t)}(\bar{\delta}_{\varepsilon^{-1}}^{c(t)} c(t + \varepsilon), \delta_{\varepsilon^{-1}}^{c(t)} c(t + \varepsilon)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

But relations (3.3.2) and (3.3.4) imply that

$$d_A^{c(t)}(\bar{\delta}_{\varepsilon^{-1}}^{c(t)} c(t + \varepsilon), \dot{c}(t)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

therefore we would finally get

$$d_A^{c(t)}(\delta_{\varepsilon^{-1}}^{c(t)} c(t + \varepsilon), \dot{c}(t)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

which is what we want to prove: that the curve c is derivable in t with respect to the dilatation structure \mathcal{A} .

Let us prove the relation (3.3.5). According to hypothesis (a') we have:

$$0 \leq \frac{1}{\varepsilon} d_A(c(t + \varepsilon), c(t)) - \frac{1}{\varepsilon} d_B(c(t + \varepsilon), c(t)) \leq \frac{1}{\varepsilon} \int_t^{t+\varepsilon} |\dot{c}(\tau)|_B \, d\tau - \frac{1}{\varepsilon} d_B(c(t + \varepsilon), c(t))$$

where the quantity

$$|\dot{c}(s)|_B = \lim_{\varepsilon \rightarrow 0} \frac{d_B((c(s + \varepsilon), c(s)))}{\varepsilon} = d_B^{c(s)}(c(s), \dot{c}(s))$$

exists for almost every $s \in [0, 1]$, according to theorem 2.11.

We obtain therefore the relation:

$$\frac{1}{\varepsilon} d_A(c(t + \varepsilon), c(t)) - \frac{1}{\varepsilon} d_B(c(t + \varepsilon), c(t)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (3.3.6)$$

Here is the moment to use the last hypothesis (d). Indeed, the relation (3.3.6) implies that

$$d_A^{c(t)}(c(t), \delta_{\varepsilon^{-1}}^{c(t)} c(t + \varepsilon)) - \frac{1}{\varepsilon} d_B^{c(t)}(c(t + \varepsilon), c(t)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (3.3.7)$$

Denote by $z(t, \varepsilon) = \delta_{\varepsilon^{-1}}^{c(t)} c(t + \varepsilon)$. The relation (3.3.7) becomes:

$$d_A^{c(t)}(c(t), z(t, \varepsilon)) - \frac{1}{\varepsilon} d_B^{c(t)}(c(t), \delta_{\varepsilon}^x z(t, \varepsilon)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0 \quad (3.3.8)$$

We also have

$$d_A^{c(t)}(c(t), z(t, \varepsilon)) = \frac{1}{\varepsilon} d_A^{c(t)}(c(t), c(t + \varepsilon)) \leq 2$$

for ε sufficiently small, because we supposed that c was reparametrized with the length. Therefore, with the notations from definition 3.6 and the paragraph following it, we have

$$\lim_{\varepsilon \rightarrow 0} z(t, \varepsilon) \in \text{Top}D(c(t))$$

From the hypothesis (d) we deduce that

$$\lim_{\varepsilon \rightarrow 0} d_A^{c(t)}(z(t, \varepsilon), Q_{\varepsilon}^{c(t)} z(t, \varepsilon)) = 0$$

Let us see what this means:

$$\lim_{\varepsilon \rightarrow 0} d_A^{c(t)}(\bar{\delta}_{\varepsilon}^{c(t)} c(t + \varepsilon), \delta_{\varepsilon}^{c(t)} c(t + \varepsilon)) = 0$$

This relation is equivalent with (3.3.5), so the proof is done. \blacksquare

4 Appendix: Dilatation structures

For the sake of completeness we list in this appendix the definition and properties of a dilatation structure, according to [3], [4].

4.1 The axioms of a dilatation structure

The axioms of a dilatation structure (X, d, δ) are listed further. The first axiom is merely a preparation for the next axioms. That is why we counted it as axiom 0.

A0. The dilatations

$$\delta_\varepsilon^x : U(x) \rightarrow V_\varepsilon(x)$$

are defined for any $\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1$. The sets $U(x), V_\varepsilon(x)$ are open neighbourhoods of x . All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is a number $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) .$$

We suppose that for all $\varepsilon \in \Gamma, \nu(\varepsilon) \in (0, 1)$, we have

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset U(x) .$$

There is a number $B \in (1, A)$ such that for any $\nu(\varepsilon) \in (1, +\infty)$ the associated dilatation

$$\delta_\varepsilon^x : W_\varepsilon(x) \rightarrow B_d(x, B) ,$$

is injective, invertible on the image. We shall suppose that $W_\varepsilon(x)$ is a open neighbourhood of x ,

$$V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in \Gamma_1$ and $u \in U(x)$ we have

$$\delta_{\varepsilon^{-1}}^x \delta_\varepsilon^x u = u .$$

We have therefore the following string of inclusions, for any $\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1$, and any $x \in X$:

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x, B) .$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

A1. We have $\delta_\varepsilon^x x = x$ for any point x . We also have $\delta_1^x = id$ for any $x \in X$.

Let us define the topological space

$$\begin{aligned} dom \delta = \{(\varepsilon, x, y) \in \Gamma \times X \times X : & \quad \text{if } \nu(\varepsilon) \leq 1 \text{ then } y \in U(x) , \\ & \text{else } y \in W_\varepsilon(x)\} \end{aligned}$$

with the topology inherited from the product topology on $\Gamma \times X \times X$. Consider also $Cl(dom \delta)$, the closure of $dom \delta$ in $\bar{\Gamma} \times X \times X$ with product topology. The function $\delta : dom \delta \rightarrow X$ defined by $\delta(\varepsilon, x, y) = \delta_\varepsilon^x y$ is continuous. Moreover, it can be continuously extended to $Cl(dom \delta)$ and we have

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^x y = x .$$

A2. For any $x, \in K$, $\varepsilon, \mu \in \Gamma_1$ and $u \in \bar{B}_d(x, A)$ we have:

$$\delta_\varepsilon^x \delta_\mu^x u = \delta_{\varepsilon\mu}^x u .$$

A3. For any x there is a function $(u, v) \mapsto d^x(u, v)$, defined for any u, v in the closed ball (in distance d) $\bar{B}(x, A)$, such that

$$\lim_{\varepsilon \rightarrow 0} \sup \left\{ \left| \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) - d^x(u, v) \right| : u, v \in \bar{B}_d(x, A) \right\} = 0$$

uniformly with respect to x in compact set.

Remark 4.1 *The "distance" d^x can be degenerated: there might exist $v, w \in U(x)$ such that $d^x(v, w) = 0$.*

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0, 1)$ such that for all $u, v \in \bar{B}_d(x, R)$ and all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0, \varepsilon_0)$, we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) .$$

With this assumption the following notation makes sense:

$$\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v .$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon^x(u, v) = \Delta^x(u, v)$$

uniformly with respect to x, u, v in compact set.

Definition 4.2 *A triple (X, d, δ) which satisfies A0, A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilatation structure.*

If the triple (X, d, δ) satisfies A0, A1, A2, A3, A4 and d^x is non-degenerate for any $x \in X$, then we call it a dilatation structure.

4.2 Tangent bundle of a dilatation structure

The following two theorems describe the most important metric and algebraic properties of a dilatation structure. As presented here these are condensed statements, available in full length as theorems 7, 8, 10 in [3].

Theorem 4.3 *Let (X, d, δ) be a dilatation structure. Then the metric space (X, d) admits a metric tangent space at x , for any point $x \in X$. More precisely we have the following limit:*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sup \{ |d(u, v) - d^x(u, v)| : d(x, u) \leq \varepsilon, d(x, v) \leq \varepsilon \} = 0 .$$

Theorem 4.4 *Let (X, d, δ) be a dilatation structure. Then for any $x \in X$ the triple $(U(x), \Sigma^x, \delta^x, d^x)$ is a normed local conical group. This means:*

- (a) Σ^x is a local group operation on $U(x)$, with x as neutral element and inv^x as the inverse element function;
- (b) the distance d^x is left invariant with respect to the group operation from point (a);
- (c) For any $\varepsilon \in \Gamma$, $\nu(\varepsilon) \leq 1$, the dilatation δ_ε^x is an automorphism with respect to the group operation from point (a);
- (d) the distance d^x has the cone property with respect to dilatations: for any $u, v \in X$ such that $d(x, u) \leq 1$ and $d(x, v) \leq 1$ and all $\mu \in (0, A)$ we have:

$$d^x(u, v) = \frac{1}{\mu} d^x(\delta_\mu^x u, \delta_\mu^x v) \quad .$$

The conical group $(U(x), \Sigma^x, \delta^x)$ can be regarded as the tangent space of (X, d, δ) at x . Further will be denoted by: $T_x X = (U(x), \Sigma^x, \delta^x)$.

By using proposition 5.4 [9] and from some topological considerations we deduce the following characterisation of tangent spaces associated to some dilatation structures. The following is corollary 4.7 [4].

Corollary 4.5 *Let (X, d, δ) be a dilatation structure with group $\Gamma = (0, +\infty)$ and the morphism ν equal to identity. Then for any $x \in X$ the local group $(U(x), \Sigma^x)$ is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a Carnot group).*

4.3 Equivalent dilatation structures

Definition 4.6 *Two dilatation structures (X, δ, d) and $(X, \bar{\delta}, \bar{d})$ are equivalent if*

- (a) *the identity map $\text{id} : (X, d) \rightarrow (X, \bar{d})$ is bilipschitz and*
- (b) *for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \bar{d} \left(\delta_\varepsilon^x u, \bar{\delta}_\varepsilon^x Q^x(u) \right) = 0, \quad (4.3.1)$$

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} d \left(\bar{\delta}_\varepsilon^x u, \delta_\varepsilon^x P^x(u) \right) = 0, \quad (4.3.2)$$

uniformly with respect to x, u in compact sets.

Proposition 4.7 *Two dilatation structures (X, δ, d) and $(X, \bar{\delta}, \bar{d})$ are equivalent if and only if*

- (a) the identity map $\text{id} : (X, d) \rightarrow (X, \bar{d})$ is bilipschitz and
- (b) for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that

$$\lim_{\varepsilon \rightarrow 0} (\bar{\delta}_\varepsilon^x)^{-1} \delta_\varepsilon^x(u) = Q^x(u), \quad (4.3.3)$$

$$\lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon^x)^{-1} \bar{\delta}_\varepsilon^x(u) = P^x(u), \quad (4.3.4)$$

uniformly with respect to x, u in compact sets.

The next theorem shows a link between the tangent bundles of equivalent dilatation structures.

Theorem 4.8 *Let (X, δ, d) and $(X, \bar{\delta}, \bar{d})$ be equivalent dilatation structures. Suppose that for any $x \in X$ the distance d^x is non degenerate. Then for any $x \in X$ and any $u, v \in X$ sufficiently close to x we have:*

$$\bar{\Sigma}^x(u, v) = Q^x(\Sigma^x(P^x(u), P^x(v))). \quad (4.3.5)$$

The two tangent bundles are therefore isomorphic in a natural sense.

4.4 Differentiable functions

Dilatation structures allow to define differentiable functions. The idea is to keep only one relation from definition 4.6, namely (4.3.1). We also renounce to uniform convergence with respect to x and u , and we replace this with uniform convergence in the "u" variable, with a conical group morphism condition for the derivative.

Definition 4.9 *Let (N, δ) and $(M, \bar{\delta})$ be two conical groups. A continuous function $f : N \rightarrow M$ is a conical group morphism if f is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.*

Definition 4.10 *Let (X, δ, d) and $(Y, \bar{\delta}, \bar{d})$ be two dilatation structures and $f : X \rightarrow Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $Q^x : T_x X \rightarrow T_{f(x)} Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that*

$$\lim_{\varepsilon \rightarrow 0} \sup \left\{ \frac{1}{\varepsilon} \bar{d} \left(f(\delta_\varepsilon^x u), \bar{\delta}_\varepsilon^{f(x)} Q^x(u) \right) : d(x, u) \leq \varepsilon \right\} = 0, \quad (4.4.6)$$

The morphism Q^x is called the derivative of f at x and will be sometimes denoted by $Df(x)$.

The function f is uniformly differentiable if it is differentiable everywhere and the limit in (4.4.6) is uniform in x in compact sets.

A trivial way to obtain a differentiable function (everywhere) is to modify the dilatation structure on the target space.

Definition 4.11 *Let (X, δ, d) be a dilatation structure and $f : (X, d) \rightarrow (Y, \bar{d})$ be a bilipschitz and surjective function. We define then the transport of (X, δ, d) by f , named $(Y, f * \delta, \bar{d})$, by:*

$$(f * \delta)_{\varepsilon}^{f(x)} f(u) = f(\delta_{\varepsilon}^x u).$$

The relation of differentiability with equivalent dilatation structures is given by the following simple proposition.

Proposition 4.12 *Let (X, δ, d) and $(X, \bar{\delta}, \bar{d})$ be two dilatation structures and $f : (X, d) \rightarrow (X, \bar{d})$ be a bilipschitz and surjective function. The dilatation structures $(X, \bar{\delta}, \bar{d})$ and $(X, f * \delta, \bar{d})$ are equivalent if and only if f and f^{-1} are uniformly differentiable.*

We shall prove now the chain rule for derivatives, after we elaborate a bit over the definition 4.10.

Let (X, δ, d) and $(Y, \bar{\delta}, \bar{d})$ be two dilatation structures and $f : X \rightarrow Y$ a function differentiable in x . The derivative of f in x is a conical group morphism $Df(x) : T_x X \rightarrow T_{f(x)} Y$, which means that $Df(x)$ is defined on a open set around x with values in a open set around $f(x)$, having the properties:

- (a) for any u, v sufficiently close to x

$$Df(x)(\Sigma^x(u, v)) = \Sigma^{f(x)}(Df(x)(u), Df(x)(v)),$$

- (b) for any u sufficiently close to x and any $\varepsilon \in (0, 1]$

$$Df(x)(\delta_{\varepsilon}^x u) = \bar{\delta}_{\varepsilon}^{f(x)}(Df(x)(u)),$$

- (c) the function $Df(x)$ is continuous, as uniform limit of continuous functions. Indeed, the relation (4.4.6) is equivalent to the existence of the uniform limit (with respect to u in compact sets)

$$Df(x)(u) = \lim_{\varepsilon \rightarrow 0} \bar{\delta}_{\varepsilon^{-1}}^{f(x)}(f(\delta_{\varepsilon}^x u)).$$

From (4.4.6) alone and axioms of dilatation structures we can prove properties (b) and (c). We can reformulate therefore the definition of the derivative by asking that $Df(x)$ exists as an uniform limit (as in point (c) above) and that $Df(x)$ has the property (a) above.

From these considerations the chain rule for derivatives is straightforward.

Proposition 4.13 *Let (X, δ, d) , $(Y, \bar{\delta}, \bar{d})$ and $(Z, \hat{\delta}, \hat{d})$ be three dilatation structures and $f : X \rightarrow Y$ a continuous function differentiable in x , $g : Y \rightarrow Z$ a continuous function differentiable in $f(x)$. Then $gf : X \rightarrow Z$ is differentiable in x and*

$$Dgf(x) = Dg(f(x))Df(x).$$

Proof. Use property (b) for proving that $Dg(f(x))Df(x)$ satisfies (4.4.6) for the function gf and x . Both $Dg(f(x))$ and $Df(x)$ are conical group morphisms, therefore $Dg(f(x))Df(x)$ is a conical group morphism too. We deduce that $Dg(f(x))Df(x)$ is the derivative of gf in x . \square

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